Lecture – 36.

Surface Area and Surface Integral :

1. If projection of the surface $z = f(x,y)$ is taken on x-y plane., then

Area Element, $\boxed{dA = \sqrt{1 + f_x^2 + f_y^2}}$

and Surface Area, $A = \iint_R dA$

$$\Rightarrow \boxed{A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy}$$

2. If projection of surface $x = g(y,z)$ is taken on y-z plane, then

Area element, $\boxed{dA = \sqrt{1 + g_y^2 + g_z^2}}$

Surface area, $\boxed{A = \iint_R \sqrt{1 + g_y^2 + g_z^2}\, dy\, dz}$

3. If projection of surface $y = h(x,z)$ is taken on x-z plane, then

Area Element, $\boxed{dA = \sqrt{1 + h_x^2 + h_z^2}}$

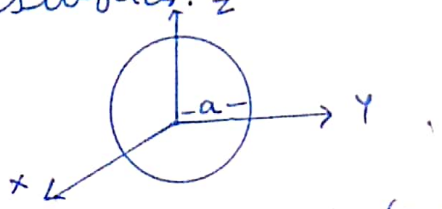
and Surface Area, $\boxed{A = \iint_R \sqrt{1 + h_x^2 + h_z^2}\, dx\, dz}$

Ex: = 15.5

Find the surface area of given surfaces.

* Here we will use short-cut (direct) formulae of for surface area of standard surfaces.

1. $$\boxed{x^2+y^2+z^2=a^2}$$



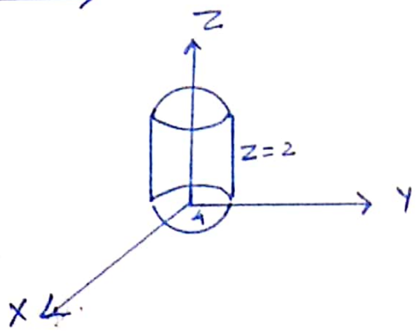
Since it is a sphere of radius 'a' and centre (0, 0, 0)

$\Rightarrow$ surface area of sphere, $S = 4\pi r^2$

$= 4\pi a^2 \qquad (\because r = a)$

$\Rightarrow \boxed{S = 4\pi a^2}$ Ans

2. $x^2+y^2=16, \quad 0 \le z \le 2$



Since it is a cylinder of height $z=2$ and radius $= 4$

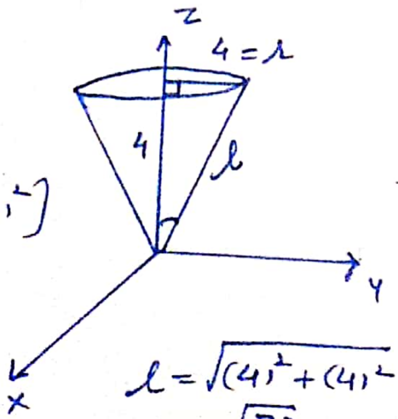
$[x^2+y^2=(4)^2]$

$\Rightarrow$ surface area of cylinder, $S = 2\pi r h$

$= 2\pi(4)(2) = 16\pi$

$\Rightarrow \boxed{S = 16\pi}$ Ans.

3. $z^2 = x^2+y^2 ; \quad 0 \le z \le 4.$



Since it is a cone with height $z=4$ and topmost radius $r=4$. $[x^2+y^2=z^2=(4)^2]$

$\Rightarrow$ surface area of cone, $S = \pi r l$

$= \pi(4)\sqrt{(4)^2+(4)^2}$

$= 4\pi(4\sqrt{2})$

$= 16\sqrt{2}\,\pi$

$\Rightarrow \boxed{S = 16\sqrt{2}\,\pi}$ Ans.

$l = \sqrt{(4)^2+(4)^2}$

$= \sqrt{32}$

$= 4\sqrt{2}$

Surface Integral

$$\text{Surface Integral} = \iint_S f(x,y,z)\,.\,dA$$

where $dA = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$ (If projection is on xy-plane)

or $dA = \dfrac{dy\,.\,dz}{\hat{n}\cdot\hat{i}}$ (If projection is on yz-plane)

or $dA = \dfrac{dz\,.\,dx}{\hat{n}\cdot\hat{j}}$ (If projection is on zx-plane)

Here $\hat{n} = \dfrac{(\text{grad } f)}{|\text{grad } f|}$

* The numericals based on 'Surface Integral' are lengthy, so have less chances for MCQ. But above written formulae can be asked as MCQ.

Flux of a vector field through a surface S.

Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ represent velocity field of a fluid. The total volume of the fluid flowing through S per unit time is called 'Flux of V through S'. It is given by:

$$\text{Flux} = \iint_S (\vec{V}\cdot\hat{n})\,dA.$$

* Ques are quite lengthy. These have no chance for MCQ.

Q. Evaluate the surface integral $\iint_S f(x,y,z)\,dA$

where $f = 6xyz$, S is the portion of the plane $x+y+z=1$ in the first octant.

Sol: Here $f = 6xyz$; Let $g: x+y+z=1$

$\Rightarrow$ grad $g = (g_x\hat{i} + g_y\hat{j} + g_z\hat{k}) = (g_x\hat{i} + g_y\hat{j} + g_z\hat{k})$

$= \cancel{(6yz)\hat{i} + (6xz)\hat{j} + (6xy)\hat{k}}$

$= (\hat{i} + \hat{j} + \hat{k})$

$$\hat{n} = \frac{\text{grad } g}{|\text{grad } g|} = \frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{1^2+1^2+1^2}}$$

$$\Rightarrow \quad \hat{n} = \frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}$$

Let projection be taken on xy-plane

$$\therefore \quad dA = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}}\cdot\hat{k}}$$

$$\Rightarrow \quad dA = \frac{dx\,dy}{1/\sqrt{3}} = \sqrt{3}\,dx\,dy$$

$\therefore$ Reqd. surface integral is:

$$I = \iint_S f(x,y,z)\,dA$$

$$= \iint_S (6xyz)\cdot\sqrt{3}\,dx\,dy$$

$$= 6\sqrt{3}\iint_S xy(1-x-y)\,dx\,dy \qquad (\because z = 1-x-y)$$

For limits:

On xy-plane $(z=0)$

$\therefore \quad x+y=1 \quad \Rightarrow \quad y = 1-x$

In first octant $\quad 0 \le x \le 1$

$\quad\quad 0 \le y \le (1-x)$

$$\therefore \quad I = 6\sqrt{3} \int_{x=0}^{1} \int_{y=0}^{(1-x)} xy(1-x-y)\,dx\,dy$$

$$= 6\sqrt{3} \int_{x=0}^{1} \int_{y=0}^{(1-x)} (xy - x^2y - xy^2)\,dx\,dy$$

$$= 6\sqrt{3} \int_{x=0}^{1} \left[\int_{y=0}^{1-x} (xy - x^2y - xy^2)\,dy \right] dx$$

$$= 6\sqrt{3} \int_{x=0}^{1} \left[x\frac{y^2}{2} - x^2\frac{y^2}{2} - x\frac{y^3}{3} \right]_{y=0}^{(1-x)} dx$$

$$= \cancel{6}\sqrt{3} \int_{x=0}^{1} \left[\frac{3xy^2 - 3x^2y^2 - 2xy^3}{\cancel{6}} \right]_{y=0}^{(1-x)} dx$$

$$= \sqrt{3} \int_{x=0}^{1} \left[3xy^2 - 3x^2y^2 - 2xy^3 \right]_{y=0}^{1-x} dx$$

$$= \sqrt{3} \int_{x=0}^{1} \underbrace{\left[3x(1-x)^2 - 3x^2(1-x)^2 - 2x(1-x)^3 \right] dx}_{\left(\frac{1}{20}\right)}$$

On simplification, we get:

$$\boxed{I = \frac{\sqrt{3}}{20}}$$

Ans